Math 211 - Bonus Exercise 3 (please discuss on Forum)

1) Consider integers m, n, a such that $m|an$ and consider the homomorphism

$$f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}, \qquad (k \bmod n) \mapsto (ak \bmod m)$$

Describe the kernel and the image of f explicitly (they will both be of the form $\mathbb{Z}/d\mathbb{Z}$ with d related to m, n, a and their various common divisors) and check that the first isomorphism theorem holds.

2) Consider the function

$$f : \mathbb{C}\backslash 0 \to \mathbb{R}\backslash 0, \qquad f(a + bi) = a^2 + b^2$$

Show that f is a homomorphism if both domain and codomain are interpreted as groups with respect to multiplication. Describe the kernel and image of f (geometrically in terms of the real line and the complex plane).

3) Let $\mathbb{F}$ be any field and consider the group of invertible upper triangular matrices with coefficients in $\mathbb{F}$ (the operation is matrix multiplication)

$$G = \left\{ \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \middle| a, b, x \in \mathbb{F}, a \neq 0, b \neq 0 \right\}$$

Consider the subset $H \subset G$ of matrices as above but with $a = b = 1$.

a) Show that H is a normal subgroup of G

b) Find a "simple" description of H, i.e. find a group that is isomorphic to H (and prove the isomorphism). Naturally, the sought-for group should be "simpler" than H, and in particular it should not involve any matrices.

c) Find a "simple" description of G/H, as explained above.

4) Consider the dihedral group D_{2n} and consider any natural number $m|n$. Let $H \leq D_{2n}$ be the subgroup generated by rotation by $\frac{2\pi m}{n}$ degrees.

a) Show that H is a normal subgroup of D_{2n}

b) Calculate the quotient subgroup D_{2n}/H (i.e. show that it is isomorphic to some "known" group).

5) Assume that a normal subgroup $N \triangleleft G$ has the property that $[G : N]$ is a prime integer. Then for any subgroup $H \leq G$, show that either $H \leq N$ or $G = HN$ (the latter option says that G is generated by H and N).

6) If H_1 and H_2 are normal subgroups of G such that $G = H_1 H_2$, prove that

$$G/(H_1 \cap H_2) \cong G/H_1 \times G/H_2$$

Show that this implies the Chinese remainder theorem.